MAL 111: INTRODUCTION TO ANALYSIS & DIFFERENTIAL EQUATIONS

MINOR 2

MAX. MARKS. 25

Throughout, $\mathbb{R}$ stands for the metric space $\mathbb{R}$ with the usual euclidean metric.

I: For each of the following metric spaces (X, d) and $A \subseteq X$, determine the interior, closure and boundary of A. [3+3 Marks]

[no justification required]

(1) Let $X = \mathcal{R}$, the set of reals with $d = D$ the discrete metric

$$D(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

and $A = \mathcal{N}$, the set of all natural numbers.

(2) Let $X = \mathbb{R}$ and $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

II: Answer the following. [5+5 Marks]

a. Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in a metric space (X, d).

Show that

$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$, for all $n, m \in \mathbb{N}$.

Hence, deduce that the sequence $\{d(x_n, y_n)\}$ converges in $\mathbb{R}$.

b. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$. Then prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

III: Answer the following. [4+5 Marks]

i. Calculate $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2} dt$.

ii. Let $f \in \mathcal{R}[a, b]$, for all $a < b \in \mathbb{R}$. Further, suppose that the improper integral $\int_0^\infty f(t)dt$ converges in $\mathbb{R}$. Show that given $\epsilon > 0$, there exists $M_\epsilon > 0$ such that

$$\left| \int_c^d f(t)dt \right| < \epsilon$$

for all $d > c \geq M_\epsilon$.

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